Journal of Mathematical Sciences: Advances and Applications
Volume 4, Number 2, 2010, Pages 439-449

NUMERICAL SOLUTION OF KAWAHARA'S EQUATION BY COMBINING HOMOTOPY PERTURBATION AND VARIATIONAL ITERATION METHODS

M. MATINFAR, M. MAHDAVI and Z. RAEISY

Department of Mathematics
Science Faculty
Mazandaran University
P. O. Box 47415-1468, Babolsar
Iran
e-mail: m.matinfar@umz.ac.ir

Abstract

In this paper, exact and numerical solutions are obtained for the Kawahara's equation by variational homotopy perturbation method (VHPM). Comparisons are made among the variational iteration method (VIM), the exact solutions and the proposed method. The results reveal that the proposed method is very effective and simple, and can be applied for other nonlinear problems in mathematical.

1. Introduction

In this paper, we consider the numerical solutions to a problem involving a nonlinear partial differential equation of the form

$$u_t + uu_x + u_{3x} - u_{5x} = 0, \tag{1}$$

which is called *Kawahara's equation*. We solve Equation (1), subject to the initial condition

2010 Mathematics Subject Classification: 35J20.

Keywords and phrases: variational homotopy perturbation method, Kawahara's equation.

Received January 13, 2010; Revised January 20, 2010

$$u(x, 0) = f(x), \quad x \in \mathbb{R}. \tag{2}$$

The Kawahara's equation occurs in the theory of magneto-acoustic waves in a plasma, and in the theory of shallow water waves with surface tension [14].

In order to solve this equation numerically, we use variational homotopy perturbation method. The numerical results are compared with the exact solutions and those, which have been previously obtained by the variational iteration method in [14].

2. Variational Homotopy Perturbation Method

To convey the basic idea of the variational homotopy perturbation method, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to variational iteration method [1-4, 7-10, 12], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\{Lu_n + N\tilde{u}_n - g(\tau)\}\, d\tau, \tag{4}$$

where $\lambda(\tau)$ is a Lagrange multiplier [1-4, 7-10, 12], which can be identified optimally via the variational iteration method. The subscripts n denote the n-th approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n = 0$ and (4) is called a correct functional. Now, we apply the homotopy perturbation method;

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + p\int_0^x \lambda(\tau)\left\{N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right\} d\tau - \int_0^x \lambda(\tau) g(\tau)\, d\tau, \tag{5}$$

which is the variational homotopy perturbation method, and is formulated by the coupling of variational iteration method and Adomian's polynomials.

The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [5, 6, 11, 13, 15]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [5, 6, 11, 13, 15] to obtain

$$f = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \cdots. \tag{6}$$

If $p \to 1$, then (6) becomes the approximate solution of the form

$$u = \lim_{p \to 1} f = u_0 + u_1 + u_2 + \cdots. \tag{7}$$

A comparison of like powers of p gives solutions of various orders.

3. VHPM for Kawahara's Equation

In this section, we consider Equation (1) with initial conditions

$$u(x, 0) = f(x),$$

and apply VHPM on it. For solving Equation (1) by using the VHPM, we consider

$$L(u) = u_t, \tag{8}$$

$$N(u) = uu_x + u_{3x} - u_{5x}, \tag{9}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [1-4, 7-10, 12], we can construct a correct functional as follows:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\tau) \{u_{n_\tau} + \tilde{u}_n \tilde{u}_{n_x} + \tilde{u}_{n_{3x}} - \tilde{u}_{n_{5x}}\} \, d\tau, \tag{10}$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \{u_{n_\tau} + u_n u_{n_x} + u_{n_{3x}} - u_{n_{5x}}\} \, d\tau. \tag{11}$$

Applying the variational homotopy perturbation method, we have:

$$u_0 + pu_1 + p^2u_2 + \cdots = f(x)$$

$$+ p\int_0^t -(u_0 + pu_1 + p^2u_2 + \cdots)(u_{0_x} + pu_{1_x} + \cdots)\, d\tau$$

$$+ p\int_0^t -(u_{0_{3x}} + pu_{1_{3x}} + p^2u_{2_{3x}} + \cdots)\, d\tau$$

$$+ p\int_0^t (u_{0_{5x}} + pu_{1_{5x}} + p^2u_{2_{5x}} + \cdots)\, d\tau. \tag{12}$$

Comparing the coefficients of like powers of p, we have:

$$p^0 : u_0(x, t) = f(x),$$

$$p^1 : u_1(x, t) = -\int_0^t u_0u_{0_x} d\tau - \int_0^t u_{0_{3x}} d\tau + \int_0^t u_{0_{5x}} d\tau,$$

$$p^2 : u_2(x, t) = -\int_0^t (u_0u_{1_x} + u_1u_{0_x})\, d\tau - \int_0^t u_{1_{3x}} d\tau + \int_0^t u_{1_{5x}} d\tau,$$

$$p^3 : u_3(x, t) = -\int_0^t (u_0u_{2_x} + u_2u_{0_x})\, d\tau - \int_0^t u_{2_{3x}} d\tau + \int_0^t u_{2_{5x}} d\tau,$$

$$\vdots.$$

Thus, the components which constitute $u(x, t)$ are written like this

$$u(x, t) = u_0 + u_1 + u_2 + \cdots.$$

For later numerical computation, we let the expression

$$\varphi_n = \sum_{i=0}^{n} u_i(x, t), \tag{13}$$

to denote the n-term approximation to $u(x, t)$.

4. Implementation of the Method

In this section, two important cases of Kawahara's equation, which correspond to some real physical processes will be investigated to show the reliability of the proposed scheme.

Example 1. We consider the Equation (1) with $u(x, 0) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4(kx)$, where $k = \frac{1}{2\sqrt{13}}$. By using the Equation (12), we have:

$$u_0(x, t) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4\left(\frac{\sqrt{13}}{26}x\right),$$

$$u_1(x, t) = \frac{-7560}{371293}t\sqrt{13}\,\frac{\sinh\left(\frac{\sqrt{13}}{26}x\right)}{\cosh^5\left(\frac{\sqrt{13}}{26}x\right)},$$

$$u_2(x, t) = \frac{68040}{62748517}t^2\,\frac{4\cosh^2\left(\frac{\sqrt{13}}{26}x\right) - 5}{\cosh^6\left(\frac{\sqrt{13}}{26}x\right)},$$

$$\vdots$$

Thus, the components which constitute $u(x, t)$ are written like this

$$u(x, t) = u_0 + u_1 + u_2 + \cdots.$$

The exact value of $u(x, t)$ in a closed form is

$$u(x, t) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4[k(x + ct)], \tag{14}$$

where $c = \frac{36}{169}$ and $k = \frac{1}{2\sqrt{13}}$,

as presented in[14].

In what follows, we present the absolute errors between φ_{1VHPM}, and the exact solution and also the absolute errors between φ_{2VHPM} and the exact solution in Tables 1 and 2, for the values of $t = 0.1(0.1)0.5$ and $x = 0.1(0.1)0.5$.

Table 1. The numerical results for φ_1 in comparison with the exact solution of u

$t_j \mid x_i$	0.1	0.2	0.3	0.4	0.5
0.1	$1.0827e-005$	$1.0781e-005$	$1.0706e-005$	$1.0603e-005$	$1.0472e-005$
0.2	$4.3296e-005$	$4.3105e-005$	$4.2799e-005$	$4.2378e-005$	$4.1846e-005$
0.3	$9.7393e-005$	$9.6945e-005$	$9.6236e-005$	$9.5273e-005$	$9.4058e-005$
0.4	$1.7310e-004$	$1.7227e-004$	$1.7098e-004$	$1.6923e-004$	$1.6704e-004$
0.5	$2.7039e-004$	$2.6904e-004$	$2.6697e-004$	$2.6420e-004$	$2.6072e-004$

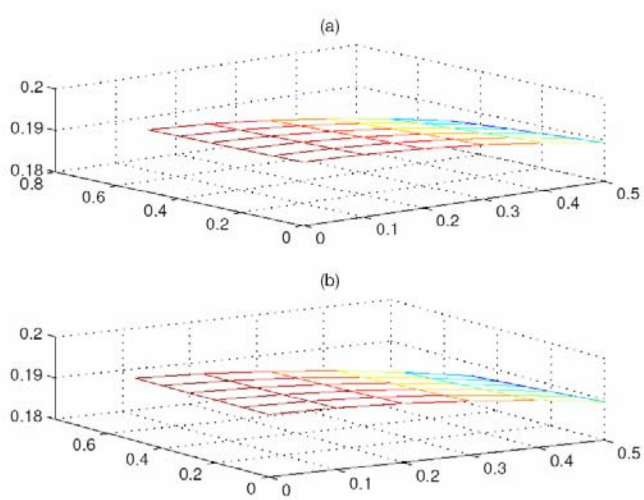

Figure 1. Comparison between the (a) $u(x, t)$, and (b) φ_{1VHPM} for the values of $t = 0(0.1)0.5$, $x = 0(0.1)0.5$.

Table 2. The numerical results for φ_2 in comparison with the exact solution of u

$t_j \| x_i$	0.1	0.2	0.3	0.4	0.5
0.1	$2.1812e-009$	$4.2408e-009$	$6.2785e-009$	$8.2840e-009$	$1.0247e-008$
0.2	$1.8330e-008$	$3.4799e-008$	$5.1089e-008$	$6.7116e-008$	$8.2801e-008$
0.3	$6.4832e-008$	$1.2039e-007$	$1.7533e-007$	$2.2936e-007$	$2.8223e-007$
0.4	$1.6071e-007$	$2.9234e-007$	$4.2246e-007$	$5.5041e-007$	$6.7554e-007$
0.5	$3.2762e-007$	$5.8457e-007$	$8.3850e-007$	$1.0881e-006$	$1.3322e-006$

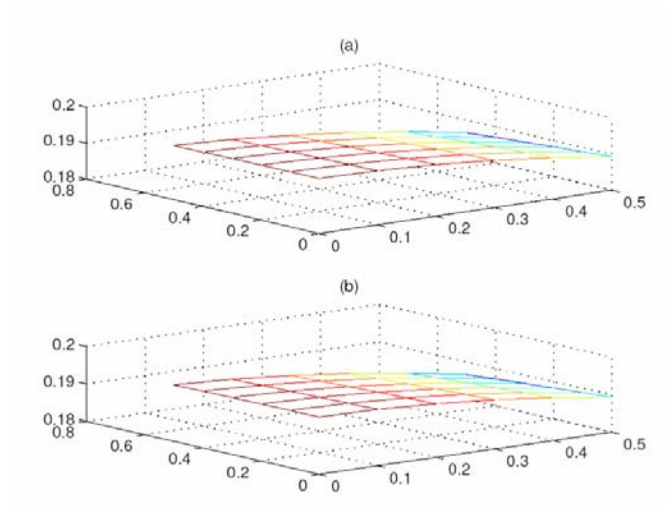

Figure 2. Comparison between the (a) $u(x, t)$, and (b) φ_{2VHPM} for the values of $t = 0(0.1)0.5$, $x = 0(0.1)0.5$.

Example 2. Now, we consider the Equation (1) with $u(x, 0) = -\frac{72}{169} + \frac{420}{169} \frac{\operatorname{sech}^2[kx]}{(1 + \operatorname{sech}^2[kx])^2}$, where $k = \frac{1}{2\sqrt{13}}$. By using the Equation (12), we have:

$$u_0(x, t) = -\frac{72}{169} + \frac{420}{169} \frac{\operatorname{sech}^2[kx]}{(1 + \operatorname{sech}^2[kx])^2},$$

$$\begin{aligned} u_1(x, t) = {} & 53 \frac{\frac{630}{371293} \cosh(kx)\sqrt{13}t \sinh(kx)}{(\cosh^2(kx) + 1)^7} \\ & - 40 \frac{\frac{630}{371293} \sqrt{13}t \cosh(kx) \sinh(kx) \cosh^{10}(kx)}{(\cosh^2(kx) + 1)^7} \\ & - 565 \frac{\frac{630}{371293} \sqrt{13}t \cosh(kx) \sinh(kx) \cosh^{2}(kx)}{(\cosh^2(kx) + 1)^7} \\ & + 951 \frac{\frac{630}{371293} \sqrt{13}t \cosh(kx) \sinh(kx) \cosh^{4}(kx)}{(\cosh^2(kx) + 1)^7} \\ & - 231 \frac{\frac{630}{371293} \sqrt{13}t \cosh(kx) \sinh(kx) \cosh^{6}(kx)}{(\cosh^2(kx) + 1)^7} \\ & + 80 \frac{\frac{630}{371293} \sqrt{13}t \cosh(kx) \sinh(kx) \cosh^{8}(kx)}{(\cosh^2(kx) + 1)^7}, \end{aligned}$$

$$\vdots$$

Thus, the components which constitute $u(x, t)$ are written like this

$$u(x, t) = u_0 + u_1 + u_2 + \cdots.$$

The exact value of $u(x, t)$ in a closed form is

$$u(x, t) = -\frac{72}{169} + \frac{420}{169} \frac{\operatorname{sech}^2[(kx + ct)]}{(1 + \operatorname{sech}^2[k(x + ct)])^2}, \tag{15}$$

where $c = \dfrac{36}{169}$ and $k = \dfrac{1}{2\sqrt{13}}$, as presented in [14].

In what follows, we present the absolute errors between φ_{1VHPM} and the exact solution in Table 3, for the values of $t = 0.1(0.1)0.5$ and $x = 0.1(0.1)0.5$.

Table 3. The numerical results for φ_1 in comparison with the exact solution of u

$t_j \mid x_i$	0.1	0.2	0.3	0.4	0.5
0.1	$6.0868e-004$	$9.4059e-004$	$1.3000e-003$	$1.6000e-003$	$1.9000e-003$
0.2	$1.8000e-003$	$2.4000e-003$	$3.1000e-003$	$3.7000e-003$	$4.7000e-003$
0.3	$3.5000e-003$	$4.5000e-003$	$5.4000e-003$	$6.4000e-003$	$7.4000e-003$
0.4	$5.7000e-003$	$7.0000e-003$	$8.3000e-003$	$9.6000e-003$	$1.0900e-002$
0.5	$8.5000e-003$	$1.0100e-002$	$1.1700e-002$	$1.3400e-002$	$1.5000e-002$

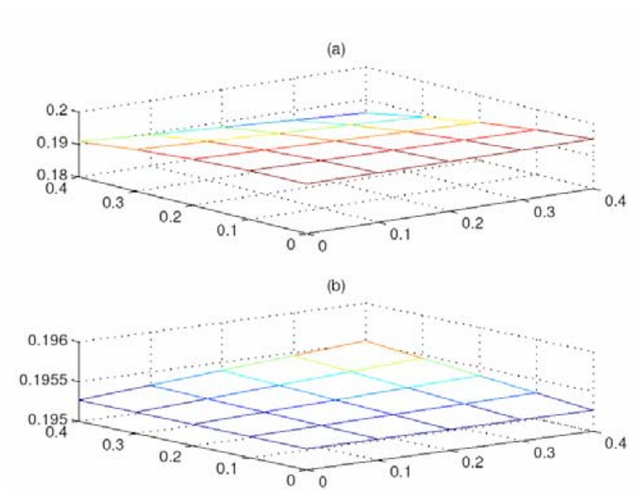

Figure 3. Comparison between the (a) $u(x, t)$, and (b) φ_{1VHPM} for the values of $t = 0(0.1)0.4$, $x = 0(0.1)0.4$.

The numerical results reveal that, the VHPM is easy to implement and reduces the computational work to a tangible level, while still maintaining a very higher level of accuracy.

5. Conclusion

In this paper, variational homotopy perturbation method was employed successfully for solving the Kawahara's equation. The small amount of computation compared to that required in other methods such as the variational iteration method, and the rapid convergence show that the method is reliable, and provides a significant improvement in solving partial differential equations over existing methods. The computations in this paper are done by MATLAB software.

References

[1] F. Z. Geng and M. G. Cui, Solving nonlinear multi-point boundary value problems by combining homotopy perturbation and variational iteration methods, Int. J. Nonlinear Sci. Num. Simul. 10 (2009), 597-600.

[2] J. H. He, Comput. Methods Appl. Mech. Eng. 167 (1998), 69.

[3] J. H. He, Variational iteration method-a kind of non-linear analytical technique: Some examples, Int. J. Nonlinear Mech. 34(4) (1999), 699-708.

[4] J. H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Eng. 178(3-4) (1999), 257-262.

[5] J. H. He, Comparison of homotopy perturbation method and homotopy analysis method, Appl. Math. Comput. 156 (2) (2004), 527-539.

[6] J. H. He, Homotopy perturbation method for bifurcation of nonlinear problems, Int. J. Nonlinear Sci. Num. Simul. 6(2) (2005), 207-208.

[7] J. H. He, Non-Perturbative Method for Strongly Nonlinear Problems, Dissertation, de Verlag in Internet GmbH, Berlin, 2006.

[8] J. H. He, Some asymptotic methods for strongly nonlinear equations, Int. J. Mod. Phys. B 20(1) (2006), 1141-1199.

[9] J. H. He, Int. J. Mod. Phys. B 20(18) (2006), 2561.

[10] J. H. He and X. H. Wu, Chaos Solitons Fractals 29 (2006), 108.

[11] J. H. He, Homotopy perturbation method for solving boundary value problems, Physics Letters A 350(1-2) (2006), 87-88.

[12] J. H. He and X. H. Wu, Variational iteration method: New development and applications, Computers and Mathematics with Applications 54(7-8) (2007), 881-894.

[13] J. H. He, G. C. Wu and F. Austin, The variational iteration method which should be followed, Nonlinear Science Letters A 1 (2010), 1-30.

[14] M. Matinfar, H. Hosseinzadeh and M. Ghanbari, Exact and numerical solution of Kawahara equation by the variational iteration method, Appl. Math. Sci. 2(43) (2008), 2119-2126.

[15] S. T. Mohyud-Din, Solving heat and wave-like equations using He's polynomials, Mathematical Problems in Engineering 2009 (2009), Doi:10.1155/2009/427516.

■